

The Cameron-Liebler problem for sets

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Abstract

Cameron-Liebler line classes and Cameron-Liebler k -classes in $\text{PG}(2k+1, q)$ are currently receiving a lot of attention. Here, links with the Erdős-Ko-Rado results in finite projective spaces occurred. We introduce here in this article the similar problem on Cameron-Liebler classes of sets, and solve this problem completely, by making links to the classical Erdős-Ko-Rado result on sets. We also present a characterisation theorem for the Cameron-Liebler classes of sets.

Keywords: Cameron-Liebler set, Erdős-Ko-Rado problem

MSC 2010 codes: 05A18, 05B25, 51E20

1 Introduction

In [4], Cameron and Liebler investigated the orbits of the projective groups $\text{PGL}(n+1, q)$. For this purpose they introduced line classes in the projective space $\text{PG}(3, q)$ with a specific property, which afterwards were called *Cameron-Liebler line classes*. A Cameron-Liebler line class \mathcal{L} with parameter x in $\text{PG}(3, q)$ is a set of $x(q^2 + q + 1)$ lines in $\text{PG}(3, q)$ such that any line $\ell \in \mathcal{L}$ meets precisely $x(q+1) + q^2 - 1$ lines of \mathcal{L} in a point and such that any line $\ell \notin \mathcal{L}$ meets precisely $x(q+1)$ lines of \mathcal{L} in a point.

Many equivalent characterisations are known, of which we present one. For an overview we refer to [7, Theorem 3.2]. A *line spread* of $\text{PG}(3, q)$ is a set of lines that form a partition of the point set of $\text{PG}(3, q)$, i.e. each point of $\text{PG}(3, q)$ is contained in precisely one line of the line spread. The lines of a line spread are necessarily pairwise skew. Now a line set \mathcal{L} in $\text{PG}(3, q)$ is a Cameron-Liebler line class with parameter x if and only if it has x lines in common with every line spread of $\text{PG}(3, q)$.

The central problem for Cameron-Liebler line classes in $\text{PG}(3, q)$, is to determine for which parameters x a Cameron-Liebler line class exists, and to classify the examples admitting a given parameter x . Constructions of Cameron-Liebler line classes and characterisation results were obtained in [3, 4, 5, 9, 13, 16]. Recently several results were obtained through a new counting technique, see [11, 12, 14]. A complete classification is however not in sight.

Also recently, Cameron-Liebler k -classes in $\text{PG}(2k+1, q)$ were introduced in [17] and Cameron-Liebler line classes in $\text{PG}(n, q)$ were introduced in [12]. Both generalise the classical Cameron-Liebler line classes in $\text{PG}(3, q)$.

Before describing the central topic of this article, we recall the concept of a q -analogue. In general a q -analogue is a mathematical identity, problem, theorem,..., that depends on a variable q and that generalises a known identity, problem, theorem,..., to which it reduces in the (right) limit $q \rightarrow 1$. In a combinatorial/geometrical setting it often arises by replacing a set and its subsets by a vector space and its subspaces. E.g. the q -binomial theorem is a q -analogue of the classical

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binomial theorem. In recent years there has been a lot of attention for q -analogues, see [1] amongst others.

The Cameron-Liebler problem has not arisen as a q -analogue of a problem on sets, but has an interesting counterpart on sets that we will describe and investigate in this article. The definition builds on the spread definition of the classical Cameron-Liebler line classes and uses a classical set counterpart for spreads in a projective space. A subset of size k of a set will be called a k -subset or shortly a k -set.

Definition 1.1. A k -uniform partition of a finite set Ω , with $|\Omega| = n$ and $k \mid n$, is a set of pairwise disjoint k -subsets of Ω such that any element of Ω is contained in precisely one of the k -subsets.

Necessarily, a k -uniform partition of a finite set Ω , with $|\Omega| = n$, contains $\frac{n}{k}$ different k -subsets. This definition now allows us to present the definition of a Cameron-Liebler class of k -sets.

Definition 1.2. Let Ω be a finite set with $|\Omega| = n$ and let k be a divisor of n . A *Cameron-Liebler class of k -sets with parameter x* is a set of k -subsets of Ω which has x different k -subsets in common with every k -uniform partition of Ω .

Note that the q -analogue of the above definition is actually a Cameron-Liebler $(k-1)$ -class in $\text{PG}(n-1, q)$, a concept that has not been discussed before, but which is a straightforward generalisation of the Cameron-Liebler classes that have already been discussed.

We present two results on these Cameron-Liebler classes of subsets. In Theorem 2.5 we show that also for Cameron-Liebler classes of subsets many equivalent characterisations can be found. The second main theorem of this paper is the following classification result.

Theorem 1.3. Let Ω be a finite set with $|\Omega| = n$ and let \mathcal{L} be a Cameron-Liebler class of k -sets with parameter x in Ω , $k \geq 2$. If $n \geq 3k$ and \mathcal{L} is nontrivial, then either $x = 1$ and \mathcal{L} is the set of all k -subsets containing a fixed element or $x = \frac{n}{k} - 1$ and \mathcal{L} is the set of all k -subsets not containing a fixed element.

2 The classification result

The next result is the Erdős-Ko-Rado theorem, a classical result in combinatorics.

Theorem 2.1 ([8, Theorem 1] and [18]). If \mathcal{S} is a family of k -subsets in a set Ω with $|\Omega| = n$ and $n \geq 2k$, such that the elements of \mathcal{S} are pairwise not disjoint, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$. Moreover, if $n \geq 2k+1$, then equality holds if and only if \mathcal{S} is the set of all k -subsets through a fixed element of Ω .

Lemma 2.2. Let Ω be a finite set with $|\Omega| = n$, and let \mathcal{L} be a Cameron-Liebler class of k -sets with parameter x in Ω , with $k \mid n$.

1. The number of k -uniform partitions of Ω equals $\frac{n!}{\left(\frac{n}{k}\right)!(k!)^{\frac{n}{k}}}$.
2. The number of k -sets in \mathcal{L} equals $x \binom{n-1}{k-1}$.
3. The set $\overline{\mathcal{L}}$ of k -subsets of Ω not belonging to \mathcal{L} is a Cameron-Liebler class of k -sets with parameter $\frac{n}{k} - x$.

Proof. 1. With every permutation (ordering) σ of the n elements of Ω , we can construct a partition P_σ in the following way: for every $i = 1, \dots, \frac{n}{k}$ the elements on the positions $(i-1)k+1, (i-1)k+2, \dots, ik$ form a k -subset of Ω , and these $\frac{n}{k}$ subsets are pairwise disjoint and form thus a k -uniform partition. Now, every partition can arise from $\left(\frac{n}{k}\right)!(k!)^{\frac{n}{k}}$ different permutations as the $\frac{n}{k}$ subsets can be permuted and each of these k -subsets can be permuted internally.

2. We perform a double counting of the tuples (C, P) , with $C \in \mathcal{L}$, P a k -uniform partition and C a k -set in P . We find that

$$|\mathcal{L}| \frac{(n-k)!}{\left(\frac{n}{k}-1\right)!(k!)^{\frac{n-k}{k}}} = x \frac{n!}{\left(\frac{n}{k}\right)!(k!)^{\frac{n}{k}}} \Rightarrow |\mathcal{L}| = x \frac{n! k}{(n-k)! k! n} = x \binom{n-1}{k-1}.$$

3. Since every k -uniform partition of Ω contains x subsets belonging to \mathcal{L} , it contains $\frac{n}{k} - x$ subsets belonging to $\overline{\mathcal{L}}$. \square

Example 2.3. Let Ω be a finite set with $|\Omega| = n$, and assume $k \mid n$. We give some examples of Cameron-Liebler classes of k -sets with parameter x . Note that $0 \leq x \leq \frac{n}{k}$.

- The empty set is obviously a Cameron-Liebler class of k -sets with parameter 0, and directly or via the last property in Lemma 2.2 it can be seen that the set of all k -subsets of Ω is a Cameron-Liebler class of k -sets with parameter $\frac{n}{k}$. These two examples are called the *trivial* Cameron-Liebler classes of k -sets.
- Let p be a given element of Ω . The set of k -subsets of Ω containing p is a Cameron-Liebler class of k -sets with parameter 1. Indeed, in every k -uniform partition of Ω there is exactly one k -subset containing p .

Again using the last property of Lemma 2.2, we find that the set of all k -subsets of Ω not containing the element p is a Cameron-Liebler class of k -sets with parameter $\frac{n}{k} - 1$.

In the introduction we already mentioned that many equivalent characterisations for Cameron-Liebler classes in $\text{PG}(3, q)$ are known. In Theorem 2.5 we show that this is also true for Cameron-Liebler classes of subsets. We did not mention the equivalent characterisations for the Cameron-Liebler sets in $\text{PG}(3, q)$, but they arise as the q -analogues of the characterisations in Theorem 2.5.

Before stating this theorem, we need to introduce some concepts. The *incidence vector* of a subset A of a set S is the vector whose positions correspond to the elements of S , with a one on the positions corresponding to an element in A and a zero on the other positions. Below we will use the incidence vector of a family of k -subsets of a set Ω : as this family is a subset of the set of all k -subsets of Ω , each position corresponds to a k -subset of Ω . For any vector v whose positions correspond to elements in a set, we denote its value on the position corresponding to an element a by $(v)_a$. The all-one vector will be denoted by j .

Given a set Ω , we also need the *incidence matrix of elements and k -subsets*. This is the $|\Omega| \times \binom{|\Omega|}{k}$ -matrix whose rows are labelled with the elements of Ω , whose columns are labelled with the k -sets of Ω and whose entries equal 1 if the element corresponding to the row is contained in the k -set corresponding to the column, and zero otherwise. The *Kneser matrix* or *disjointness matrix* of k -sets in Ω is the $\binom{|\Omega|}{k} \times \binom{|\Omega|}{k}$ -matrix whose rows and columns are labelled with the k -sets of Ω and whose entries equal 1 if the k -set corresponding to the row and the k -set corresponding to the column are disjoint, and zero otherwise¹.

We will need the following result about the Kneser matrix.

Lemma 2.4. Let Ω be a finite set with $|\Omega| = n$ and let K be the Kneser matrix of the k -sets in Ω . The eigenvalues of K are given by $\lambda_j = (-1)^j \binom{n-k-j}{k-j}$, $j = 0, \dots, k$, and the multiplicity of the eigenvalue λ_j is $\binom{n}{j} - \binom{n}{j-1}$.

A direct but lengthy proof of this result can be found in [15]. The original proofs of this result can be found in [6, Theorem 4.6] and [19, Section 2(d)], but both use the theory of association schemes, which we did not introduce in this article. The Kneser matrix is a matrix of the Johnson scheme.

Now we can present a theorem with many equivalent characterisations of Cameron-Liebler classes of k -subsets.

¹This Kneser graph is often introduced as the adjacency matrix of the disjointness graph, but as there is no need here to introduce this graph, we introduced it directly.

Theorem 2.5. Let Ω be a finite set with $|\Omega| = n$, and let k be a divisor of n . Let \mathcal{L} be a set of k -subsets of Ω with incidence vector χ . Denote $\frac{|\mathcal{L}|}{\binom{n-1}{k-1}}$ by x . Let C be the incidence matrix of elements and k -subsets in Ω and let K be the Kneser matrix of k -sets in Ω . The following statements are equivalent.

- (i) \mathcal{L} is a Cameron-Liebler class of k -sets with parameter x .
- (ii) \mathcal{L} has x different k -subsets in common with every k -uniform partition of Ω .
- (iii) For each fixed k -subset π of Ω , the number of elements of \mathcal{L} disjoint from π equals $(x - (\chi)_\pi) \binom{n-k-1}{k-1}$.
- (iv) The vector $\chi - \frac{kx}{n}j$ is contained in the eigenspace of K for the eigenvalue $-(\binom{n-k-1}{k-1})$.
- (v) $\chi \in \text{row}(C)$.
- (vi) $\chi \in (\ker(C))^\perp$.

Proof. If $k = n$, then there is only one k -set in Ω . This one k -set could be contained in \mathcal{L} (case $x = 1$) or not (case $x = 0$), but in both cases all the statements are valid. Note that K is the 1×1 zero matrix. From now on we assume that $n \geq 2k$.

By Definition 1.2 statements (i) and (ii) are equivalent. Now, we assume that statement (ii) is valid. Let π be a fixed k -subset of Ω and denote the number of elements of \mathcal{L} disjoint from π by N . Double counting the set of tuples (σ, U) , with σ a k -set of $\Omega \setminus \pi$ and U a k -uniform partition of Ω containing π and σ , we find

$$\begin{aligned} N \frac{(n-2k)!}{\left(\frac{n-2k}{k}\right)!(k!)^{\frac{n-2k}{k}}} &= \frac{(n-k)!}{\left(\frac{n-k}{k}\right)!(k!)^{\frac{n-k}{k}}} (x - (\chi)_\pi) \\ \Leftrightarrow N &= (x - (\chi)_\pi) \frac{(n-k)!}{\left(\frac{n-2k}{k}\right)!(k!)^{\frac{n-2k}{k}}} \frac{k}{n-k} = (x - (\chi)_\pi) \binom{n-k-1}{k-1}, \end{aligned}$$

since U contains x elements of \mathcal{L} disjoint from π if $\pi \notin \mathcal{L}$, and $x - 1$ elements of \mathcal{L} disjoint from π if $\pi \in \mathcal{L}$.

If statement (iii) is valid for the set \mathcal{L} , then it follows immediately that $K\chi = \binom{n-k-1}{k-1}(xj - \chi)$ since on the position corresponding to the k -set π the vector $K\chi$ has the number of elements in \mathcal{L} disjoint from π ; the vector $\binom{n-k-1}{k-1}(xj - \chi)$ has $\binom{n-k-1}{k-1}(x - (\chi)_\pi)$ on this position. We also know that $Kj = \binom{n-k}{k}j$ since for every k -set in Ω there are $\binom{n-k}{k}$ different k -sets disjoint to it. We find that

$$\begin{aligned} K \left(\chi - \frac{kx}{n}j \right) &= \binom{n-k-1}{k-1}(xj - \chi) - \frac{kx}{n} \binom{n-k}{k}j \\ &= -\binom{n-k-1}{k-1} \left(\chi - xj + \frac{n-k}{k} \frac{kx}{n}j \right) = -\binom{n-k-1}{k-1} \left(\chi - \frac{kx}{n}j \right). \end{aligned}$$

Hence, $\chi - \frac{kx}{n}j$ is a vector in the eigenspace of K for the eigenvalue $-\binom{n-k-1}{k-1}$.

We assume that statement (iv) is valid. Let V be the eigenspace of K related to the eigenvalue $-\binom{n-k-1}{k-1}$. Then $\chi \in \langle j \rangle \oplus V$. By Example 2.3 the set of all k -sets containing a fixed element p is a Cameron-Liebler set with parameter 1. Denote the incidence vector of this Cameron-Liebler set by v_p . Then v_p^t is the row of C corresponding to p . Since (i) implies (iv), through (ii) and (iii), we know that $v_p^t - \frac{k}{n}j$ is a vector in the eigenspace of K for the eigenvalue $-\binom{n-k-1}{k-1}$. So $v_p^t \in \langle j \rangle \oplus V$. It follows that $\text{row}(C)$ is a subspace of $\langle j \rangle \oplus V$.

The set of all k -sets of Ω forms a $k - (n, k, 1)$ block design, hence also a $2 - (n, k, \binom{n-2}{k-2})$ block design, whose incidence matrix is C . From Fisher's inequality (see [2, 10]) it follows that $\text{rk}(C) = n$. Using Lemma 2.4 we know that $\dim(\langle j \rangle \oplus V) = n$. Since $\text{row}(C) \subseteq \langle j \rangle \oplus V$ and

$\dim(\text{row}(C)) = n = \dim(\langle j \rangle \oplus V)$, the subspaces $\text{row}(C)$ and $\langle j \rangle \oplus V$ are equal. Consequently, $\chi \in \text{row}(C)$.

Statements (v) and (vi) are clearly equivalent as $\text{row}(C) = (\ker(C))^\perp$.

Finally, we assume that statement (vi) is valid. Let U be a k -uniform partition of Ω and let χ_U be its incidence vector. It is clear that $C\chi_U = j$ since each element of Ω is contained in precisely one element of U . Since $Cj = \binom{n-1}{k-1}j$ it follows that $\chi_U - \frac{1}{\binom{n-1}{k-1}}j \in \ker(C)$. From the assumption that (vi) is valid, it follows that χ and $\chi_U - \frac{1}{\binom{n-1}{k-1}}j$ are orthogonal. Hence,

$$|\mathcal{L} \cap U| = \langle \chi, \chi_U \rangle = \frac{1}{\binom{n-1}{k-1}} \langle \chi, j \rangle = \frac{|\mathcal{L}|}{\binom{n-1}{k-1}} = x,$$

which proves (ii). \square

The main theorem of this paper, Theorem 1.3, states that the examples in Example 2.3 are the only examples of Cameron-Liebler classes of k -sets, in case $n \geq 3k$. We note that only four parameter values are admissible. The next lemmata show this result.

Lemma 2.6. Let \mathcal{L} be a nontrivial Cameron-Liebler class of k -sets with parameter x in a set Ω of size $n \geq 3k$, $x < \frac{n}{k} - 1$ and $k \geq 2$. Then, \mathcal{L} is the set of all k -sets through a fixed element and $x = 1$.

Proof. It follows immediately from the definition of a Cameron-Liebler class of k -sets with parameter x that there are x pairwise disjoint k -sets in \mathcal{L} . Let $H_1, \dots, H_x \in \mathcal{L}$ be x pairwise disjoint k -sets, and let \mathcal{S}_i be the set of k -sets in \mathcal{L} which are disjoint to $H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_x$, $i = 1, \dots, x$. It is clear that \mathcal{S}_i is a set of k -sets in $\Omega_i = \Omega \setminus (H_1 \cup \dots \cup H_{i-1} \cup H_{i+1} \cup \dots \cup H_x)$ and $|\Omega_i| = n - (x-1)k$. Actually \mathcal{S}_i is a Cameron-Liebler class of k -sets with parameter 1 in Ω_i . Hence, the size of \mathcal{S}_i equals $\binom{n-(x-1)k-1}{k-1}$ by the second property in Lemma 2.2. Moreover, the elements of \mathcal{S}_i mutually intersect. So, we apply Theorem 2.1 for the set \mathcal{S}_i of k -subsets of Ω_i . Since $n - (x-1)k > 2k \Leftrightarrow x < \frac{n}{k} - 1$, we know that $\mathcal{S}_i \subseteq \mathcal{L}$ is the set of all k -sets through a fixed element $p_i \in \Omega_i$. Necessarily, $p_i \in H_i$.

We prove that all k -sets in \mathcal{L} pass through at least one of the elements p_j , $1 \leq j \leq x$. Assume that $H' \in \mathcal{L}$ and $p_j \notin H'$, for all $j = 1, \dots, x$. Denote $|H' \cap H_i|$ by k_i , $i = 1, \dots, x$. We know that $k' = \sum_{i=1}^x k_i \leq k$ since the sets $H_1, \dots, H_x \in \mathcal{L}$ are pairwise disjoint. Since $|\Omega \setminus (H' \cup (\cup_{i=1}^x H_i))| = n - (x+1)k + k' \geq k + k'$, we can find a k' -set J in $\Omega \setminus (H' \cup (\cup_{i=1}^x H_i))$. Let $\{J_i \mid i = 1, \dots, x\}$ be a partition of J , such that $|J_i| = k_i$. The sets $H'_i = (H_i \setminus H') \cup J_i$, for $i = 1, \dots, x$, are pairwise disjoint k -sets. Moreover, $p_i \in H_i \setminus H' \subseteq H'_i$ for all $i = 1, \dots, x$. Since H'_i and $H_1 \cup \dots \cup H_{i-1} \cup H_{i+1} \cup \dots \cup H_x$ are disjoint, and $p_i \in H'_i$, the set H'_i belongs to \mathcal{L} . However, then H'_1, \dots, H'_x, H' are $x+1$ pairwise disjoint k -sets in \mathcal{L} , a contradiction. Hence, all elements of \mathcal{L} are k -sets through one of the elements p_j , $j = 1, \dots, x$.

There are $\binom{n-1}{k-1}$ different k -sets through p_j , $j = 1, \dots, x$. If $x \geq 2$, then there are k -sets containing at least two of the elements p_j , $j = 1, \dots, x$, since $k \geq 2$, and hence the total number of k -sets containing one of the elements p_j , $j = 1, \dots, x$, is smaller than $x\binom{n-1}{k-1}$. However, all k -sets in \mathcal{L} contain at least one of the elements p_j , $j = 1, \dots, x$, and $|\mathcal{L}| = x\binom{n-1}{k-1}$. We find a contradiction. So, $x = 1$ and in this case, \mathcal{L} consists of all k -sets through p_1 . \square

Lemma 2.7. Let \mathcal{L} be a Cameron-Liebler class of k -sets with parameter $\frac{n}{k} - 1$ in a set Ω of size $n \geq 3k$, with $k \geq 2$. Then, \mathcal{L} is the set of all k -sets not through a fixed element.

Proof. By the last property of Lemma 2.2, the set $\overline{\mathcal{L}}$ of k -subsets of Ω not belonging to \mathcal{L} is a Cameron-Liebler class of k -sets with parameter 1. By Lemma 2.6 $\overline{\mathcal{L}}$ is the set of all k -subsets of Ω containing a fixed element p . Consequently, \mathcal{L} is the set of all k -subsets of Ω not containing p . \square

Proof of Theorem 1.3. This result combines the results of Lemma 2.6 and Lemma 2.7. \square

We end this paper with the discussion of a few cases that are not covered by the main theorem.

Remark 2.8. Let Ω be a set of size n , and let k be a divisor of n . Theorem 1.3 does not cover the cases $k = 1$, and $n \in \{k, 2k\}$.

- Assume $k = 1$, then any set of x different 1-subsets of Ω is a Cameron-Liebler class of k -sets with parameter x . So, in this case each value x , with $0 \leq x \leq n$, is admissible as parameter of a Cameron-Liebler class.
- If $n = k$, there is only one subset of size k , and thus all Cameron-Liebler classes of k -sets are trivial.
- If $n = 2k$, each k -uniform partition consists of two k -sets which are the complement of each other. Every set of k -subsets that is constructed by picking one of both k -sets from each k -uniform partition, is a Cameron-Liebler class of k -sets with parameter 1, equivalently, it is an Erdős-Ko-Rado set. There are $2^{\binom{2k-1}{k-1}}$ different choices to pick $\binom{2k-1}{k-1}$ different k -sets, but many choices give rise to isomorphic examples. For $k = 1, 2, 3$, there are 1, 2 and 11 nonisomorphic examples, respectively.

Acknowledgment: The research of Maarten De Boeck is supported by the BOF-UGent (Special Research Fund of Ghent University). The research of Andrea Švob has been partially supported by the Croatian Science Foundation under the project 1637 and has been supported by a grant from the COST project *Random network coding and designs over $GF(q)$* (COST IC-1104). The authors thank the referees for their suggestions for improvements to the original version of this article.

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